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Wavelet-Type Decompositions and Approximations from Shift-Invariant Spaces

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We investigate sufficient conditions on principal shift-invariant spaces $S(\phi)$ in order to provide prescribed approximation orders in $L_p(\mathbb{R}^d)$, $1<p<\infty$. Our results are applicable even in pathological cases where $\hat{\phi}(0)=0$ and are based on the wavelet-type decompositions of Besov spaces. © 1997 Academic Press

1. INTRODUCTION

A space of functions S is called shift-invariant if it is invariant under all integer translates (**shifts**), i.e.,

$$f \in S \quad \Leftrightarrow \quad f(\cdot+\alpha) \in S, \qquad \forall \alpha \in \mathbb{Z}^d.$$

We will focus our attention to the so-called **principal shift-invariant** subspaces (**PSI**) $S := S(\phi)$, of $L_p := L_p(\mathbb{R}^d)$, $1<p<\infty$, that are generated by the closure of the linear span of the shifts of ϕ in the topology of L_p.

To unlock the approximation potential of a SI (shift invariant) space we consider the stationary scale of spaces S^h, $h>0$,

$$S^h := S^h(\phi) := \{f(\cdot/h) : f \in S\}.$$

It is widely accepted that the qualitative critarion by which we rate the efficiency for approximation of such spaces is their *approximation order*. Roughly speaking, we say that the scale of spaces S^h provides order of approximation $r>0$ in L_p, $1<p<\infty$, if for every *sufficiently smooth* function f the error of approximation from S^h satisfies

$$\inf_{s \in S^h} \|f-s\|_{L_p} = O(h^r), \qquad h \to 0, \tag{1.1}$$

where the constant in O is independent of h.

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Our goal is to obtain weak assumptions on ϕ so that (1.1) holds. First we have to address what we really mean by *sufficiently smooth*. In the literature one will not find a unique answer and there is a good reason for this; there is a trade-off between the size of the set of smooth functions that can be approximated with order r on the one hand, and on the other, the conditions that one needs to impose on ϕ in order to achieve the approximation. So, one way of weakening the assumptions on ϕ is to reduce the set of the approximable functions X [BR, J2, R]. Traditionally X was taken to be the Sobolev space W_p^r, while for our purposes the role of X will be played by the Besov spaces $B_p^{r,q}$, $0<q\leqslant 1$ (see definitions below).

DEFINITION 1.2. Let $1<p<\infty$ and $r>0$. We say that the shift-invariant space S provides order of approximation r in $L_p(\mathbb{R}^d)$ if for every function $f\in B_p^{r,q}$, $0<q\leqslant 1$, the error of approximation from S satisfies

$$\inf_{s\in S^h} \|f-s\|_{L_p} \leqslant \text{const}\, h^r\, \|f\|_{\dot{B}_p^{r,q}}, \qquad h\to 0,$$

where the above constant is independent of f and h.

Over the past years the problem of determining the approximation orders of SI spaces has been basically approached in two different ways. The first is the traditional one in which the approximation schemes as well as the error analysis are given at the time domain and they generally exploit the fact that polynomials enjoy locally good approximation properties. These methods have been associated with the so-called Strang–Fix conditions

$$\text{(i)}\quad \hat{\phi}(0)\neq 0,$$

and (1.3)

$$\text{(ii)}\quad D^\alpha\hat{\phi}(2\pi v)=0 \qquad \text{for all}\quad |\alpha|<r \quad\text{and}\quad v\in\mathbb{Z}^d\backslash\{0\},$$

and the approximation schemes that are employed are given by what are called **quasi-interpolant** operators. These operators exercise the fact that under certain decay assumptions on ϕ, the Strang–Fix conditions guarantee that the space $\Pi_{<r}$ (of polynomials of degree $<r$) lies in $\bigcap_h S^h$. This approach was initiated by Schoenberg [S] and has been followed by several mathematicians [BJ, CJW, DM, LC, LJ, SF] (for complete references see [BR]).

The second approach was initiated by de Boor and Ron [BR] (for $p=\infty$) and has been extended to various settings in [BDR, K1, K2, R, J1, J2, Z]. In these references the approximation is taking place in the frequency domain and the techniques employed have their roots in harmonic analysis. The main advantage of the latter methods is that by and large they require no a priori assumptions on ϕ.

However, the common denominator in the above references (with the exception of [BDR, R, Z] which treat the case $p=2$) is that $\hat{\phi}(0) \neq 0$; it is one of the goals of our present work to circumvent this condition. We will do this by means of the wavelet-type decompositions of various function spaces.

Wavelet-type decompositions present themselves as a new powerful tool for analyzing the approximation properties of SI spaces and allow one to decompose a function f into basic building blocks $\{\psi_{v,j}\}_{(v,j) \in \mathbb{Z} \times \mathbb{Z}^d}$, called **wavelets** (or sometimes **atoms**), that enjoy superior smoothness and decay properties, i.e.,

$$f(\cdot) = \sum_{v \in \mathbb{Z}} \sum_{j \in \mathbb{Z}^d} b_{v,j} \psi_{v,j}(\cdot). \tag{1.4}$$

Moreover, the functions $\{\psi_{v,j}\}_{(v,j) \in \mathbb{Z} \times \mathbb{Z}^d}$ are generated by means of a single function ψ, that is,

$$\psi_{v,j}(\cdot) := \psi(2^v \cdot - j). \tag{1.5}$$

The idea is to construct the approximation in two different phases. In the first, which does not really involve the **SI** space, we approximate, with the right accuracy, the function f with a truncated version of its wavelet decomposition $f^{v(h)}$,

$$f^{v(h)}(\cdot) := \sum_{v \leqslant v(h)} \sum_{j \in \mathbb{Z}^d} b_{v,j} \psi_{v,j}(\cdot),$$

and in the second phase, using the approximation potential of the SI space, we approximate individually each wavelet $\psi_{v,j}$ with an element $s^h_{v,j} \in S^h$.

Our approximation scheme is then given by

$$s^{v(h)}(\cdot) := \sum_{v \leqslant v(h)} \sum_{j \in \mathbb{Z}^d} b_{v,j} s^h_{v,j}(\cdot).$$

One of the advantages of this method is that, in effect, it disengages the approximation properties of SI spaces from particular smoothness spaces since one only needs to approximate ideal functions, namely $\psi_{v,j}$, which can be chosen depending on the application.

These ideas have been used before in [K3] and in [J2]. In [K3] we used building blocks that were sufficiently smooth and compactly supported while in [J2] they were required to have compactly supported spectrum (i.e., the support of the Fourier transform). Here, similarly to [J2], we will concentrate in decompositions generated by functions in $\widehat{C_0^\infty}$ (the space of all functions that the Fourier transform maps in C_0^∞). However, our initial analysis will hold for more general wavelet-type decompositions.

One of the main characteristic properties of wavelets, which we are going to take advantage of, is that they have zero mean value. This will basically allow us to use approximants whose Fourier transform vanishes at the origin. On the other hand, this is also the reason for restricting p to $1<p<\infty$ because it is known that such decompositions do not hold for $p=1,\ \infty$, in the framework of L_p spaces.

Throughout this paper, we shall use standard multi-index notation; for every $x=(x_1, ..., x_d)\in\mathbb{R}^d$ and $\alpha=(\alpha_1, ..., \alpha_d)\in\mathbb{N}^d$, we define $x^\alpha := x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, $|\alpha| := \alpha_1+\cdots+\alpha_d$, and $D^\alpha := \partial^{|\alpha|}/(\partial^{\alpha_1}x_1\cdots\partial^{\alpha_d}x_d)$.

By $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ we denote the Schwartz space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^d$ and by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions. The Fourier transform $\hat{f}$ of a summable function is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e_\xi(-x)\, dx,$$

where $e_\xi(\cdot) := e^{i(\cdot)\xi}$.

We shall also denote by Π the space of all polynomials on $\mathbb{R}^d$ and with $\mathcal{S}'/\Pi$ the space of equivalence classes of distributions in $\mathcal{S}'$ modulo polynomials.

Let now $\psi\in\mathcal{S}$ with the following properties:

$$\begin{aligned} &\text{(i)}\quad \operatorname{supp}\hat{\psi}\subset\{2^{-1}\leqslant|\xi|\leqslant 2\}\\ &\text{(ii)}\quad |\hat{\psi}(\xi)|\geqslant \text{const}>0 \qquad \text{if}\quad 3/5\leqslant\xi\leqslant 5/3,\\ &\text{(iii)}\quad \sum_{v\in\mathbb{Z}}|\hat{\psi}(2^v\xi)|^2=1 \qquad \text{if}\quad \xi\neq 0. \end{aligned} \tag{1.6}$$

We set $\psi_v(\cdot) := 2^{vd}\psi(2^v)$, $v\in\mathbb{Z}$. For $0<r<\infty$, $1\leqslant p\leqslant\infty$, and $0<q\leqslant\infty$, the homogeneous Besov space $\dot{B}_p^{r,q}$ is defined to be the set of all $f\in\mathcal{S}'/\Pi$ such that

$$\|f\|_{\dot{B}_p^{r,q}} := \|(2^{vr}\,\|\psi_v * f\|_{L_p})_v\|_{l_q(\mathbb{Z})} \tag{1.7}$$

is finite.

An equivalent form of (1.7) is also given by (see [FJ])

$$\|f\|_{\dot{B}_p^{r,q}} \approx \|(2^{vr}\,\|(2^{-vd/p}\,|\psi_v * f(j2^{-v})|)_j\|_{l_p(\mathbb{Z}^d)})_v\|_{l_q(\mathbb{Z})}. \tag{1.8}$$

We note also that the homogeneous Besov spaces $\dot{B}_p^{r,q}$ owe part of their name to the fact that the seminorm $\|\cdot\|_{\dot{B}_p^{r,q}}$ enjoys the following property (see [T]):

$$\|f(t\cdot)\|_{\dot{B}_p^{r,q}} \leqslant \text{const}\, t^{r-d/p} \|f\|_{\dot{B}_p^{r,q}}, \qquad t>0. \tag{1.9}$$

By adding the norm $\|f\|_{L_p}$ we get the inhomogeneous Besov space $B_p^{r,q}$ equipped with

$$\|f\|_{B_p^{r,q}} := \|f\|_{L_p} + \|f\|_{\dot{B}_p^{r,q}}.$$

For any domain $\Omega \subset \mathbb{R}^d$, $1 \leqslant p \leqslant \infty$, and $m \in \mathbb{N}$ we denote by $W_p^m(\Omega)$ the Sobolev space defined by

$$W_p^m(\Omega) := \left\{ f\colon \|f_{W_p^m(\Omega)} := \sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{L_p(\Omega)} < \infty \right\}.$$

For $1 \leqslant p \leqslant \infty$ we define $\mathscr{L}_p := \mathscr{L}_p(\mathbb{R}^d)$ to be the set of all functions f such that the norm

$$\|f\|_{\mathscr{L}_p} := \left\| \sum_{\alpha \in \mathbb{Z}^d} |f(\cdot - \alpha)| \right\|_{L_p([0,1]^d)}$$

is finite. The embedding $l_1(\mathbb{Z}^d) \subset l_p(\mathbb{Z}^d)$ gives that

$$\mathscr{L}_p(\mathbb{R}^d) \subset L_p(\mathbb{R}^d), \qquad 1 \leqslant p \leqslant \infty,$$

while $\mathscr{L}_1 = L_1$.

Finally for $1 < p < \infty$ we define q by

$$q := \begin{cases} 2, & 1 < p \leqslant 2 \\ (1 - 1/p)^{-1}, & 2 \leqslant p < \infty, \end{cases} \tag{1.10}$$

and for every $x > 0$ we denote by $\lceil x \rceil$ the least integer greater than x.

2. WAVELET-TYPE APPROXIMATION

In this section we are going to carry out both phases of our approximation. We will present two main results. The first result is going to concern more general wavelet-type decompositions, while in the second result we will focus our attention on decompositions associated with wavelets in $\widehat{C_0^\infty}$. We note also that the results presented in this section hold for general shift-invariant spaces, not only for principal ones.

DEFINITION 2.1. We say that a family of functions $\Psi := \{\psi_{\nu,j}\}_{(\nu,j)\in\mathbb{Z}\times\mathbb{Z}^d}$, generated by a singleton function $\psi \in B_p^{r,q}$, is admissible for $B_p^{r,q}$ if it satisfies the following three conditions:

(a) For every $f \in B_p^{r,q}$ there are coefficients $\{b_{\nu,j}\}_{(\nu,j)\in\mathbb{Z}\in\mathbb{Z}^d}$ such that

$$f(\cdot) = \sum_{\nu\in\mathbb{Z}} \sum_{j\in\mathbb{Z}^d} b_{\nu,j}\psi_{\nu,j}(\cdot), \qquad \text{in } L_p.$$

(b) There exist constants such that for every $f \in B_p^{r,q}$,

$$\text{const } \|f\|_{\dot{B}_p^{r,q}} \leqslant \|(2^{\nu r} \|(2^{-\nu d/p} b_{\nu,j})_j\|_{l_p(\mathbb{Z}^d)})_\nu\|_{l_q(\mathbb{Z})} \leqslant \text{const } \|f\|_{\dot{B}_p^{r,q}}.$$

(c) If $f^{\nu_0} := \sum_{\nu\leqslant\nu_0} \sum_{j\in\mathbb{Z}^d} b_{\nu,j}\psi_{\nu,j}$, $\nu_0 \in \mathbb{N}$, then,

$$\|f - f^{\nu_0}\|_{L_p} \leqslant \text{const } 2^{-\nu_0 r} \|f\|_{\dot{B}_p^{r,q}}.$$

In the literature one will find a variety of admissible families, ranging from functions in $\widehat{C_0^\infty}$ to compactly supported ones. For instance, the ϕ-transform of Frazier and Jawerth gives rise to admissible families in $\widehat{C_0^\infty}$, while on the other hand the booming field of wavelets contains paradigms of admissible families in C_0^k with k arbitrarily large. In this paper we concentrate at the ϕ-transform mainly because we want to study the approximation orders of shift-invariant spaces generated by functions with mean value zero.

Our first result reads as follows:

THEOREM 2.2. *Let* $1 < p < \infty$, $0 < q \leqslant 1$, *and* $r > 0$. *Assume also that there exists a family* Ψ, *generated by a function* ψ, *which is admissible for* $B_p^{r,q}$ *and such that*

$$\operatorname{dist}(\psi, S^h, \mathscr{L}_p) \leqslant \text{const } h^r, \qquad h \to 0. \tag{2.3}$$

Then, for every $f \in B_p^{r,q}$,

$$\operatorname{dist}(f, S^h, L_p) \leqslant \text{const } h^r \|f\|_{\dot{B}_p^{r,q}}, \qquad h \to 0. \tag{2.4}$$

Of course, for every normed space $(X, \|\cdot\|)$, we have adopted the notation

$$\operatorname{dist}(f, S^h, X) := \inf_{s\in S^h} \|f - s\|_X.$$

Proof of Theorem 2.2. We note that it is sufficient to establish (2.4) only for dyadic values of h, i.e., $h = 2^{-\nu_0}$, where ν_0 is a sufficiently large natural number. Indeed, by letting $v(h) \in \mathbb{N}$ be such that $1/2 < h2^{v(h)} \leqslant 1$, it easily follows by change of variables that

$$\begin{aligned}\operatorname{dist}(f, S^h, L_p) &\leqslant \text{const}\, \operatorname{dist}(f(h2^{v(h)}\cdot), S^{2^{-v(h)}}, L_p)\\ &\leqslant \text{const}\, 2^{-v(h)r}\, \|f(h2^{v(h)}\cdot)\|_{\dot{B}_p^{r,q}}\\ &\leqslant \text{const}\, h^r\, \|f\|_{\dot{B}_p^{r,q}},\end{aligned}$$

where in the last inequality we used the homogeneity (see (1.9)) of $\|\cdot\|_{\dot{B}_p^{r,q}}$.

Thus, without loss of generality we assume that $h = 2^{-v_0}$, $v_0 \in \mathbb{N}$. Moreover, if $\Psi := \{\psi_{v,j}\}$, $(\psi_{v,j}(\cdot) := \psi(2^v\cdot - j))$ is an admissible family we can assume that for every $v \in \mathbb{N}$ there exists $s_v \in S$ such that

$$\|\psi - s_v(2^v\cdot)\|_{\mathscr{L}_p} \leqslant \text{const}\, 2^{-vr}.$$

From (c) of Definition 2.1 we also have

$$\left\| f - \sum_{v \leqslant v_0} \sum_{j \in \mathbb{Z}^d} b_{v,j}\psi_{v,j} \right\|_{L_p} \leqslant \text{const}\, 2^{-v_0 r}\, \|f\|_{\dot{B}_p^{r,q}}. \tag{2.5}$$

Therefore, it suffices to approximate $f^{v_0} := \sum_{v \leqslant v_0} \sum_{j \in \mathbb{Z}^d} b_{v,j}\psi_{v,j}$ from $S^{2^{-v_0}}$ with the correct order. This suggests the approximation scheme

$$s^{v_0} = \sum_{v \leqslant v_0} \sum_{j \in \mathbb{Z}^d} b_{v,j} s_{v,j}^{v_0},$$

where $s_{v,j}^{v_0}(\cdot) := s_{v_0 - v}(2^{v_0}\cdot - 2^{v_0 - v} j)$. It is easily seen that $s^{v_0} \in S^{2^{-v_0}}$. Moreover,

$$\begin{aligned}\|f^{v_0} - s^{v_0}\|_{L_p} &= \left\| \sum_{v \leqslant v_0} \sum_{j \in \mathbb{Z}^d} b_{v,j}[\psi_{v,j} - s_{v,j}^{v_0}] \right\|_{L_p}\\ &\leqslant \sum_{v \leqslant v_0} \left\| \sum_{j \in \mathbb{Z}^d} b_{v,j}[\psi_{v,j} - s_{v,j}^{v_0}] \right\|_{L_p}\\ &\leqslant 2^{-vd/p} \sum_{v \leqslant v_0} \|(b_{v,j})_j\|_{\ell_p}\, \|\psi - s_{v_0 - v}(2^{v_0 - v}\cdot)\|_{\mathscr{L}_p}\\ &\leqslant \text{const}\, 2^{-v_0 r} \sum_{v \leqslant v_0} 2^{vr}\, \|(2^{-vd/p} b_{v,j})_j\|_{\ell_p}\\ &\leqslant \text{const}\, 2^{-v_0 r}\, \|f\|_{\dot{B}_p^{r,1}}\\ &\leqslant 2^{-v_0 r}\, \|f\|_{\dot{B}_p^{r,q}},\end{aligned} \tag{2.6}$$

where in the second inequality we used that $\|\sum_{j \in \mathbb{Z}^d} a_j g(\cdot - j)\|_{L_p} \leqslant \|a\|_{\ell_p} \|g\|_{\mathscr{L}_p}$, $\forall g \in \mathscr{L}_p$, $(a) \in \ell_p$ (see [JM]) and in the last the embedding $\ell_q \hookrightarrow \ell^1$, $0 < q \leqslant 1$.

Finally, employing the triangular inequality,

$$\begin{aligned}\|f-s^{\nu_0}\|_{L_p} &\leqslant \|f-f^{\nu_0}\|_{L_p}+\|f^{\nu_0}-s^{\nu_0}\|_{L_p}\\ &\leqslant \text{const}\, 2^{-\nu_0 r}\, \|f\|_{\dot{B}_p^{r,q}}. \quad \blacksquare\end{aligned}$$

Next we consider a function ψ that satisfies (1.6). Our goal is to prove that $\Psi := \{\psi_{\nu,j}\}_{(\nu,j)\in\mathbb{Z}\times\mathbb{Z}^d}$ is an admissible family. It is well known that for $f\in L_p$, $1<p<\infty$,

$$f(\cdot)=\sum_{\nu\in\mathbb{Z}} \psi_\nu * \tilde{\psi}_\nu * f(\cdot),$$

where $\tilde{\psi}(\cdot) := \overline{\psi(-\cdot)}$. Furthermore, it was proved in [FJ, Lemma 2.1] that

$$\psi_\nu * \tilde{\psi}_\nu * f(\cdot)=\sum_{j\in\mathbb{Z}^d} \tilde{\psi}_\nu * f(j2^{-\nu})\, \psi(2^\nu\cdot -j), \tag{2.7}$$

which implies that

$$f(\cdot)=\sum_{\nu\in\mathbb{Z}}\sum_{j\in\mathbb{Z}^d} \tilde{\psi}_\nu * f(j2^{-\nu})\, \psi(2^\nu\cdot -j). \tag{2.8}$$

(The convergence of the above series is considered in the sense of L_p.)

Thus, taking into account (1.8) and (2.8), Ψ will be an admissible family if we can prove condition (c) of Definition 2.1.

For this, we let $f^{\nu_0} := \sum_{\nu\leqslant\nu_0} \psi_\nu * \tilde{\psi}_\nu * f$. It follows easily that

$$\begin{aligned}\|f-f^{\nu_0}\|_{L_p} &= \Big\|\sum_{\nu>\nu_0} \psi_\nu * \tilde{\psi}_\nu * f\Big\|_{L_p}\\ &\leqslant 2^{-\nu_0 r}\sum_{\nu>\nu_0} 2^{\nu r}\, \|\psi_\nu * \tilde{\psi}_\nu * f\|_{L_p}\\ &\leqslant \text{const}\, 2^{-\nu_0 r}\sum_{\nu>\nu_0} 2^{\nu r}\, \|\psi_\nu * f\|_{L_p}\\ &\leqslant \text{const}\, 2^{-\nu_0 r}\, \|f\|_{\dot{B}_p^{r,q}}\\ &\leqslant \text{const}\, 2^{-\nu_0 r}\, \|f\|_{\dot{B}_p^{r,q}}.\end{aligned} \tag{2.9}$$

Adapting Theorem 2.2 to this specific admissible family we have the following theorem:

THEOREM 2.10. *Let $1<p<\infty$, $0<q\leqslant 1$, and $r>0$. Let also S be a shift-invariant space. If for every $\eta\in\mathcal{S}$ with* $\operatorname{supp}\hat{\eta}\subset\Omega := \{2^{-1}\leqslant|\xi|\leqslant 2\}$,

$$\operatorname{dist}(\eta, S^h, \mathcal{L}_p)\leqslant \text{const}\, h^r, \qquad h\to 0. \tag{2.11}$$

then, for every $f \in B_p^{r,q}$,

$$\operatorname{dist}(f, S^h, L_p) \leqslant \text{const } h^r \|f\|_{\dot{B}_p^{r,q}}, \qquad h \to 0.$$

The natural thing to do next is to investigate inequality (2.11) and to impose assumptions on ϕ so that the PSI space $S := S(\phi)$ will provide order of approximation r.

3. THE STRANG–FIX CONDITIONS

The approximation properties of PSI subspaces of L_2 have been completely characterized by de Boor *et al.* [BDR]. Using the Hilbert-space structure of L_2 and the isometry of the Fourier transform, they achieved a breakthrough proving the following theorem:

THEOREM 3.1. *Let* $\phi \in L_2(\mathbb{R}^d)$. *Then, the principal shift-invariant space* S *provides approximation order* $r > 0$, *in* $L_2(\mathbb{R}^d)$, *if and only if*

$$|\cdot|^{-r} \left(1 - \frac{|\hat{\phi}|^2}{[\hat{\phi}, \hat{\phi}]}\right)^{1/2}$$

is in $L_\infty([-\pi, \pi]^d)$, *where* $[\hat{\phi}, \hat{\phi}] := \sum_{\beta \in 2\pi\mathbb{Z}^d} |\hat{\phi}(\cdot + \beta)|^2$.

However, we should caution the reader that the definition of approximation orders in [BDR] deviates slightly from ours, since they are defined in terms of the potential space W_2^r instead of the Besov space $\dot{B}_p^{r,q}$.

Refining this theorem Zhao [Z] proved:

THEOREM 3.2. *Let* $\phi \in L_2$; S *provides order of approximation* $r > 0$ *if and only if there exists a constant and a neighborhood of the origin* Δ *on which*

$$\sum_{\beta \in \mathbb{Z}^d \backslash 0} |\hat{\phi}(\xi + 2\pi\beta)|^2 \leqslant \text{const } |\xi|^{2r} |\hat{\phi}(\xi)|^2.$$

For the next corollary we will need the fractional Sobolev Spaces $W_2^{\varrho}(\Delta)$ where $\varrho > 0$ and Δ is an open subset of $\mathbb{R}^d$. For their definition we refer the reader to [A].

COROLLARY 3.3. *Let* $\phi \in L_2$, $r \in \mathbb{N}$, *and* $0 \leqslant k < r$. *Let also* $A := \bigcup_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} (\Delta + \beta)$, *where* Δ *is some open cube centered at the origin. If* $\hat{\phi} \in W_2^{\varrho}(A)$ *for some* $\varrho > r + d/2$ *satisfies*

$$D^{\gamma}\hat{\phi}=0 \qquad on \quad 2\pi\mathbb{Z}^d\backslash 0 \qquad for\ all \quad |\gamma|<r, \tag{3.4}$$

$$0<\text{const}\leqslant|\hat{\phi}(\xi)|\,|\xi|^{-k}, \qquad for \quad \xi\in\Delta, \tag{3.5}$$

then S provides approximation order $r-k$.

Proof. It follows from (3.4) that for every $\beta\in 2\pi\mathbb{Z}^d\backslash 0$,

$$|\hat{\phi}(\xi+\beta)|\leqslant\text{const}\,|\xi|^r\max_{|\gamma|=r}\|D^{\gamma}\hat{\phi}\|_{L_\infty(\Delta_\beta)}, \qquad \text{for} \quad \xi\in\Delta, \tag{3.6}$$

where $\Delta_\beta:=\Delta+\beta$. From the Sobolev embedding theorem (see [A, p. 217]), since $\varrho>r+d/2$, we know that

$$\max_{0\leqslant|\gamma|\leqslant r}\|D^{\gamma}\hat{\phi}\|_{L_\infty(\Delta_\beta)}\leqslant\text{const}\,\|\hat{\phi}\|_{W_2^\varrho(\Delta_\beta)}$$

for some constant independent of β. Employing (3.5) in (3.6) we obtain that

$$\text{const}\,\frac{|\hat{\phi}(\xi+\beta)|}{|\xi|^{r-k}\,|\hat{\phi}(\xi)|}\leqslant\frac{|\hat{\phi}(\xi+\beta)|}{|\xi|^r}\leqslant\text{const}\,\|\hat{\phi}\|_{W_2^\varrho(\Delta_\beta)}, \qquad \xi\in\Delta, \quad \beta\in 2\pi\mathbb{Z}^d\backslash 0.$$

From the last inequality one easily derives, using the set-subadditivity of $\|\cdot\|^2_{W_2^\varrho(\Delta)}$ (see [A]) that

$$\sum_{\beta\in\mathbb{Z}^d\backslash 0}|\hat{\phi}(\xi+2\pi\beta)|^2\leqslant\text{const}\,|\xi|^{2(r-k)}\,|\hat{\phi}(\xi)|^2\,\|\hat{\phi}\|^2_{W_2^\varrho(\Delta)},$$

and the result follows in view of Theorem 3.2. ∎

It is obvious that in order to achieve better orders of approximation one should seek the smallest k that satisfies (3.5).

We will try to extend the above corollary to $1<p<\infty$ by means of Theorem 2.10. We will give sufficient assumptions on ϕ so that

$$\operatorname{dist}(\eta, S^h, \mathscr{L}_p)\leqslant\text{const}\,h^r, \qquad h\to 0,$$

where η is any function in $\mathscr{S}$ with

$$\operatorname{supp}\hat{\eta}\subset\Omega=\{\xi: 1/2\leqslant|\xi|\leqslant 2\}. \tag{3.7}$$

For this purpose we are going to employ the same approximation scheme $T_h(\eta)$ with [BR, K1, K2, J2], defined by

$$\widehat{T_h(\eta)}:=\sum_{\alpha\in\mathbb{Z}^d}\frac{\hat{\eta}(\xi+2\pi\alpha/h)}{\hat{\phi}(h\xi+2\pi\alpha)}\,\hat{\phi}(h\xi).$$

First we verify that $T_h(\eta)\in S^h$.

LEMMA 3.8. *Let* $\phi \in L_1 \cap L_p$ *and* $h \in (0..1]$. *If* $\hat{\phi} \neq 0$ *on* $\{\xi : h/2 \leqslant |\xi| \leqslant 2h\}$, *then* $T_h(\eta) \in S^h$.

Proof. Since $\hat{\phi}$ does not vanish on $\{\xi : h/2 \leqslant |\xi| \leqslant 2h\}$ it follows from Wiener's Lemma (see [Ru]) that

$$f := \left(\frac{\hat{\eta}(\xi)}{\hat{\phi}(h\xi)}\right)^{\vee} \in L_1.$$

Moreover, $\hat{f}$ is continuous on its support Ω. The $2\pi/h$-periodic extension of $\hat{f}$ is given by

$$\tau(\xi) := \sum_{\alpha \in \mathbb{Z}^d} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)}$$

whose Fourier series representation is

$$h^d \sum_{j \in \mathbb{Z}^d} f(hj)\, e^{-ijh\xi}.$$

Moreover, since $\hat{f}$ is compactly supported, one can show that $\|(f(hj))\|_{\ell_1} \approx \|f\|_{L_1}$ (for a detailed proof see [J2]). By Fourier inversion now it follows that

$$T_h(\eta) = \sum_{j \in \mathbb{Z}^d} f(hj)\, \phi(x/h - j),$$

which is an element of S^h since

$$\|T_h(\eta)\|_{L_p} \leqslant \text{const}\ \|(f(hj))\|_{\ell_1}\ \|\phi\|_{L_p}. \qquad \blacksquare$$

A version of the following theorem was given in [J2].

THEOREM 3.9. *Let* $1 \leqslant p \leqslant \infty$ *and* η *be in* C^∞ *with* $\operatorname{supp} \eta \subset \Omega$. *If* $0 < h \leqslant 1$ *and* $\phi \in L_1 \cap L_p$ *is such that* $\hat{\phi}$ *does not vanish on* $\{\xi : h/2 \leqslant |\xi| \leqslant 2h\}$, *then*

$$\operatorname{dist}(\eta, S^h, \mathscr{L}_p) \leqslant \left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W^m_\infty} \|\hat{\phi}(h\cdot)\|_{W^m_q(\Omega + h^{-1}2\pi\mathbb{Z}^d \backslash 0)}, \tag{3.10}$$

where m *is an integer with* $m > d/q$ (*see* (1.10)).

Proof. For $p \leqslant 2$ using the embedding $\mathscr{L}_2 \hookrightarrow \mathscr{L}_p$ we get

$$\begin{aligned}
\operatorname{dist}(\eta, S^h, \mathscr{L}_p) &\leqslant \left\| \left(\hat{\eta}(\xi) - \sum_{\alpha \in \mathbb{Z}^d} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} \right\|_{\mathscr{L}_p} \\
&= \left\| \left(\sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} \right\|_{\mathscr{L}_p} \\
&\leqslant \text{const} \left\| \left(\sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} \right\|_{\mathscr{L}_p}.
\end{aligned}$$

Invoking the inequality $\|f\|_{\mathscr{L}_p} \leqslant \text{const}\, \|(1+|\cdot|)^m f\|_{L_p}$, $m > d/q$, we get that

$$\begin{aligned}
\operatorname{dist}(\eta, S^h, \mathscr{L}_p) &\leqslant \text{const} \left\| (1+|x|)^m \left(\sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} (x) \right\|_{L_2} \\
&\leqslant \text{const} \left\| \sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right\|_{W_2^m} \\
&\leqslant \text{const} \left\| \sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \right\|_{W_\infty^m} \|\hat{\phi}(h\cdot)\|_{W_2^m(\Omega + h^{-1}2\pi\mathbb{Z}^d \setminus 0)} \\
&= \text{const} \left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W_\infty^m} \|\hat{\phi}(h\cdot)\|_{W_2^m(\Omega + h^{-1}2\pi\mathbb{Z}^d \setminus 0)}.
\end{aligned}$$

Similarly for $2 \leqslant p$ we use the well-known inequality $\|\hat{f}\|_{L_p} \leqslant \text{const}\, \|f\|_{L_q}$, to derive

$$\begin{aligned}
\operatorname{dist}(\eta, S^h, \mathscr{L}_p) &\leqslant \left\| \left(\sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} \right\|_{\mathscr{L}_p} \\
&\leqslant \text{const} \left\| (1+|x|)^m \left(\sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right)^{\vee} (x) \right\|_{L_p} \\
&\leqslant \text{const} \left\| \sum_{\alpha \in 2\pi\mathbb{Z}^d \setminus 0} \frac{\hat{\eta}(\xi + 2\pi\alpha/h)}{\hat{\phi}(h\xi + 2\pi\alpha)} \hat{\phi}(h\xi) \right\|_{W_q^m} \\
&\leqslant \text{const} \left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W_\infty^m} \|\hat{\phi}(h\cdot)\|_{W_q^m(\Omega + h^{-1}2\pi\mathbb{Z}^d \setminus 0)}. \quad \blacksquare
\end{aligned}$$

Next we will estimate the size of the right-hand side of (3.10) by isolating the zeroes of ϕ at the origin.

LEMMA 3.11. *Let P be either a homogeneous polynomial of degree k which vanishes only at the origin or $P(\cdot) = |\cdot|^k$, $k \in \mathbb{N}$. Let also ϕ be such that*

$$\hat{\phi}(\cdot) = P(\cdot)\, g(\cdot) \tag{3.12}$$

in a neighborhood of the origin Δ where $g \in C^m(\Delta)$, $m > d/q$, and $g(0) \neq 0$. Then

$$\left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W_\infty^m} \leqslant \text{const } h^{-k}, \qquad h \to 0.$$

Proof. It follows from the assumptions that there exists a neighborhood of the origin Ω_1 such that $g(\xi) \neq 0$, $\forall \xi \in \Omega_1$. For h sufficiently small, say $0 < h \leqslant h_0$, $g(h\xi) \neq 0$ on Ω. Thus,

$$\begin{aligned} \left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W_\infty^m} &= h^{-k} \left\| \frac{\hat{\eta}(\xi)/P(\xi)}{g(h\xi)} \right\|_{W_\infty^m} \\ &\leqslant \text{const } h^{-k}. \qquad \blacksquare \end{aligned}$$

Finally, to find an upper bound for $\|\hat{\phi}(h\cdot)\|_{W_q^m(\Omega + h^{-1} 2\pi \mathbb{Z}^d \backslash 0)}$ we are going to use the following lemma from [J2].

LEMMA 3.13. *Let $1 < p < \infty$ and $r \in \mathbb{N}$. Let also $\hat{\phi} \in W_q^{\varrho}(\varepsilon\Delta + 2\pi\mathbb{Z}^d \backslash 0)$ where $\varepsilon > 0$ and $\varrho := \lceil r + d/q \rceil$. If ϕ satisfies*

$$D^\gamma \hat{\phi} = 0 \qquad \text{on} \quad 2\pi\mathbb{Z}^d \backslash 0 \qquad \text{for all} \quad |\gamma| < r \tag{3.14}$$

then, for $m := \lceil d/q \rceil$,

$$\|\hat{\phi}(h\cdot)\|_{W_q^m(\Omega + h^{-1} 2\pi\mathbb{Z}^d \backslash 0)} = O(h^r), \qquad h \to 0.$$

Proof. See [J2, Theorem 7.3]. $\blacksquare$

THEOREM 3.15. *Let $1 < p < \infty$ and $r, k \in \mathbb{N}$ with $r > k$. We define $m := \lceil d/q \rceil$, $\varrho := \lceil r + d/q \rceil$, and we assume that Δ is a neighborhood of the origin and P is as in Lemma 3.11. If $\phi \in L_1 \cap L_p$ satisfies:*

(i) $\hat{\phi} \in W_q^{\varrho}(\Delta + 2\pi\mathbb{Z}^d \backslash 0)$,

(ii) $\hat{\phi}(\cdot) = P(\cdot)\, g(\cdot)$ *in Δ where $g \in C^m(\Delta)$ and $g(0) \neq 0$,*

(iii) $D^\gamma \hat{\phi} = 0$ *on $2\pi\mathbb{Z}^d \backslash 0$ for all $|\gamma| < r$,*

then S^h provides order of approximation $r - k$ in $L_p(\mathbb{R}^d)$.

Proof. According to Theorem 2.10 it is sufficient to prove that

$$\operatorname{dist}(\eta, S^h, \mathscr{L}_p) \leqslant \text{const}\, h^{r-k}, \qquad h \to 0. \tag{3.16}$$

However, from Theorem 3.9 the left side of this inequality is dominated by

$$\left\| \frac{\hat{\eta}}{\hat{\phi}(h\cdot)} \right\|_{W^m_\infty} \|\hat{\phi}(h\cdot)\|_{W^m_q(\Omega + h^{-1}2\pi\mathbb{Z}^d \backslash 0)}.$$

Employing now Lemmas 3.11 and 3.13, the theorem follows. ∎

We note that similar results (with $k = 0$) have been given by Johnson [J2, Theorems 7.3 and 7.4]. However, there exists a fundamental difference between the results of Johnson and Theorem 3.15; the analysis in [J2] does not allow the treatment of cases where $\hat{\phi}(0) = 0$.

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